Deriving Models of Change with Interpretable Parameters: Linear Estimation with Nonlinear Inference

Ethan M. McCormick
Department of Methodology & Statistics
e.m.mccormick@fsw.leidenuniv.nl

Education Statistics and Data Science
emccorm@udel.edu

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| Theoretical Statements and Mathematical Objects

Substantive hypotheses are often framed as high-level theoretical statements – e.g., "early adversity delays school achievement" or "our intervention de-couples the association between stress and achievement" – which can be difficult to match to a specific statistical model.

| Theoretical Statements and Mathematical Objects

Substantive hypotheses are often framed as high-level theoretical statements – e.g., "early adversity delays school achievement" or "our intervention de-couples the association between stress and achievement" – which can be difficult to match to a specific statistical model.

- compounded by standardized linear parameter models which limit flexible model building
- my research seeks to develop models that are directly linked to theoretical questions

Deriving models of change with interpretable parameters: Linear estimation with nonlinear inference

Ethan M. McCormick*[1]

[1]Methodology & Statistics Department, Institute of Psychology, Leiden University, Leiden, Netherlands

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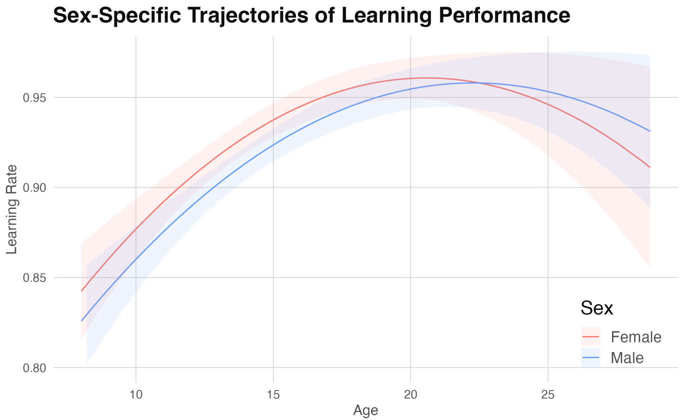
Abstract

In the modeling of change over time, there is often a disconnect between developmental theories advanced in substantive research and statistical models specified in longitudinal analysis. That is, theory is understood and advanced in terms of meaning-

| Sex-Specific Delays in Learning?



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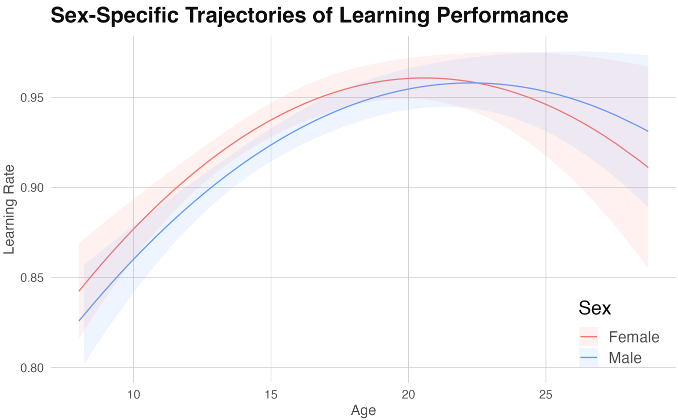

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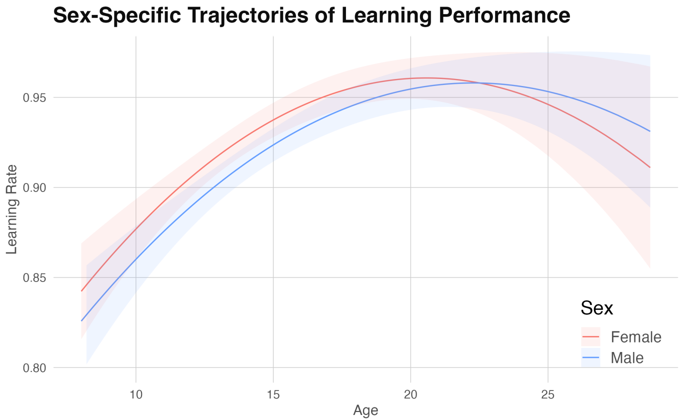
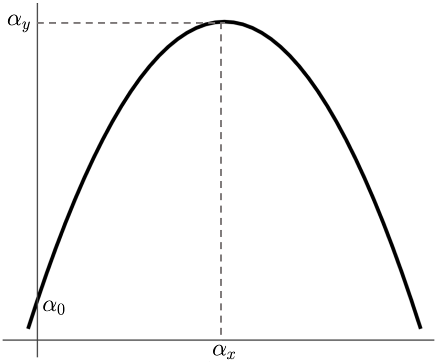


$$y_{ti} = \beta_0 + \beta_1 x_{ti} + \beta_2 x_{ti}^2 + \beta_3 x_{ti}^3$$

None of these parameters directly model the peak of the curve

| Defining Nonlinear Equations



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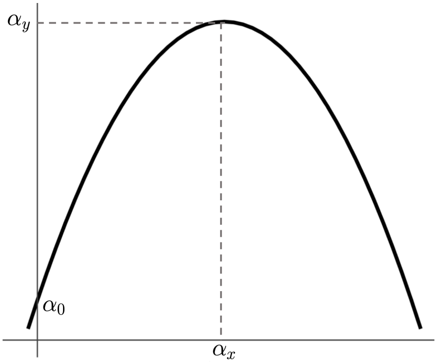

$$y_{ti} = \alpha_y - \left(\alpha_y - \alpha_0\right)\left(\frac{x_{ti}}{\alpha_0} - 1\right)^2$$

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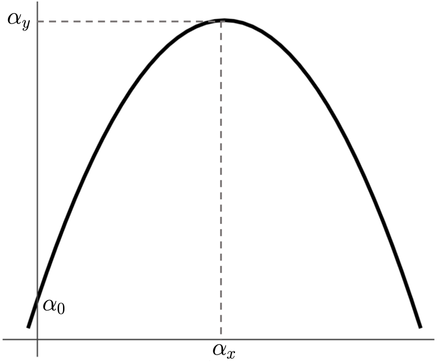

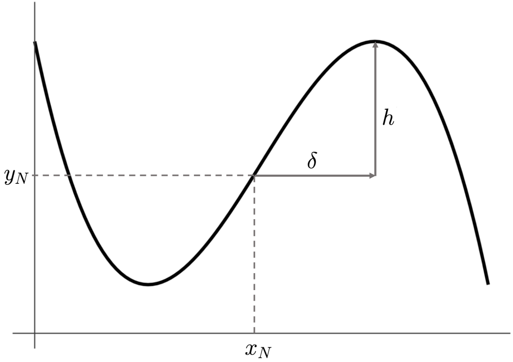

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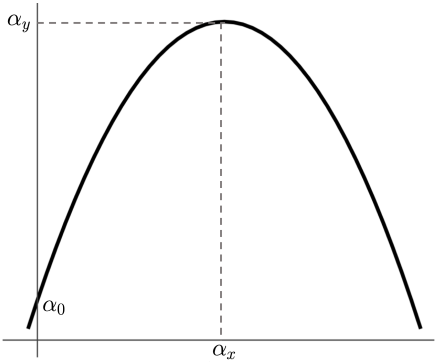

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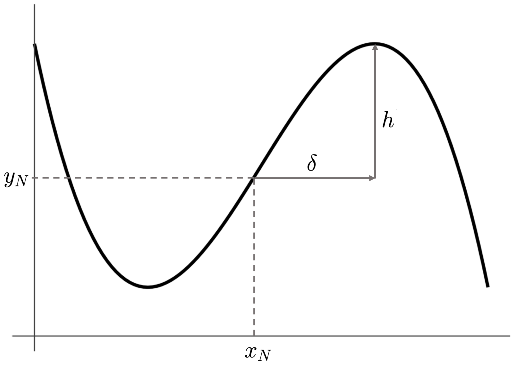
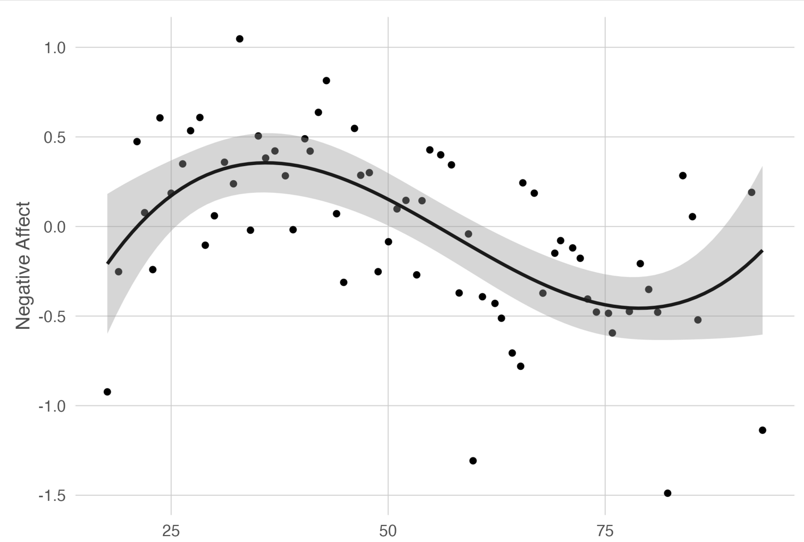


$$y_{ti} = y_N - \frac{h}{2}\left[\left(\frac{x_{ti} - x_N}{\delta}\right)^3 - 3\left(\frac{x_{ti} - x_N}{\delta}\right)\right]$$

| Negative Affect Across the Adult Lifespan

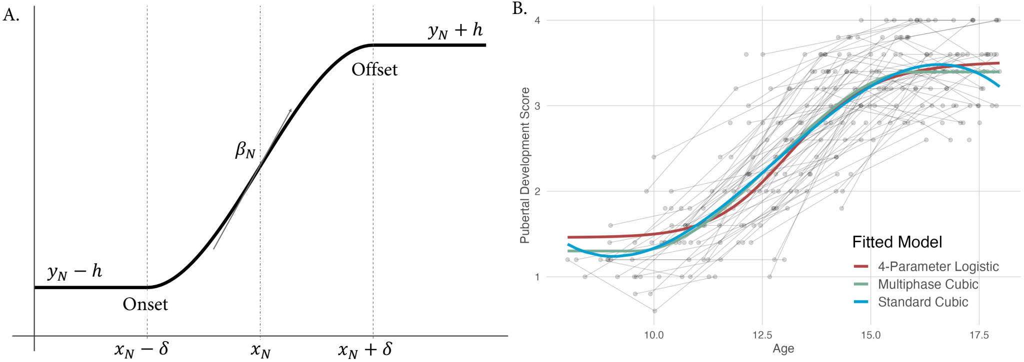


Negative Affect: Linear and Nonlinear Parameters

Fitting Linear and Nonlinear Parameter Cubic Models

	Linear Parameter Model		Nonlinear Parameter Model
β_0	-2.288^{**} (0.802)	x_N	57.428^{***} (2.162)
β_1	0.174^{**} (0.052)	y_N	-0.050 (0.051)
β_2	-0.004^{***} (0.001)	δ	21.508^{***} (1.711)
β_3	$2.04 \times 10^{-5**}$ (6.10×10^{-6})	h	-0.406^{***} (0.068)
Num.Obs.	69		69
R^2	0.355		
AIC	76.6		76.6
BIC	87.7		87.7

| Interpretable Parameters Help Define More Interesting Models



Benefits of Interpretable Parameters

Testing meaningful – and specifically articulated – theoretical hypotheses about change over time

- Timing of inflections (e.g., peaks/troughs/plateaus) [6], time-to-criterion [5], acceleration [4]

Incorporating predictors of change [3, 7]

Investigating distal outcomes associated with individual differences in change over time [8]

| Why are Nonlinear Models Not the Default?

Not defined for all values of the parameters

$$y_{ti} = \alpha_y - \left(\alpha_y - \alpha_0\right)\left(\frac{x_{ti}}{\alpha_0} - 1\right)^2$$

$$y_{ti} = y_N - \frac{h}{2}\left[\left(\frac{x_{ti} - x_N}{\delta}\right)^3 - 3\left(\frac{x_{ti} - x_N}{\delta}\right)\right]$$

| Why are Nonlinear Models Not the Default?

No hierarchy of parameters for random effects

$$y_{ti} = \beta_0 + \beta_1 x_{ti} + \beta_2 x_{ti}^2 + \beta_3 x_{ti}^3$$

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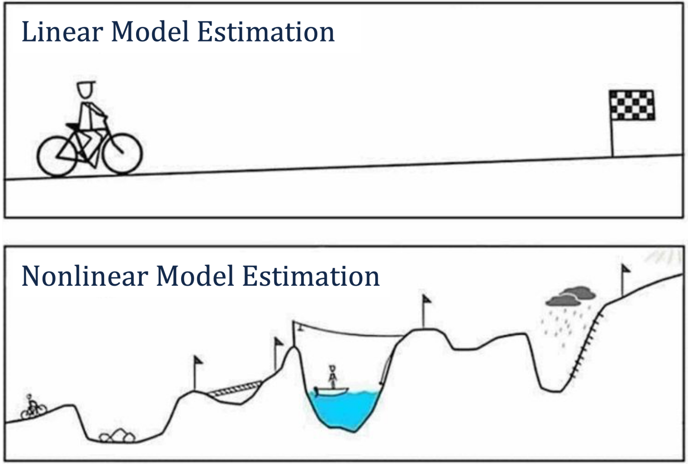

| Solutions: Linear Estimation, Nonlinear Inference (LENI)

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<u>Fixed Effects</u>

$$x_N = \frac{-\beta_2}{3\beta_3} \quad y_N = \beta_0 - \frac{\beta_1\beta_2}{3\beta_3} + \frac{2\beta_2^3}{27\beta_3^2} \quad \delta = \frac{\sqrt{\beta_2^2 - 3\beta_3\beta_1}}{3\beta_3} \quad h = -2\beta_3\delta$$

$$\text{ACOV}\left(f(x_N, y_N, \delta, h)\right) \approx \mathbf{J}'_{f(x_N,y_N,\delta,h)}\ \text{ACOV}(\boldsymbol{\beta})\ \mathbf{J}_{f(x_N,y_N,\delta,h)}$$

$$y_N - \frac{h}{2}\left[\left(\frac{x_{ti} - x_N}{\delta}\right)^3 - 3\left(\frac{x_{ti} - x_N}{\delta}\right)\right]$$

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Random Effects

$$\mathbf{T}_{f(x_N, y_N, \delta, h)} \approx \mathbf{J}'_{f(x_N, y_N, \delta, h)} \, \mathbf{T}_{\boldsymbol{\beta}} \, \mathbf{J}_{f(x_N, y_N, \delta, h)}$$

$$\text{ACOV}\left(\mathbf{T}_{f(x_N, y_N, \delta, h)}\right) \approx \mathbf{J}'_{\mathbf{T}_{f(x_N, y_N, \delta, h)}} \, \text{ACOV}\left(\mathbf{T}_{\boldsymbol{\beta}}\right) \, \mathbf{J}_{\mathbf{T}_{f(x_N, y_N, \delta, h)}}$$

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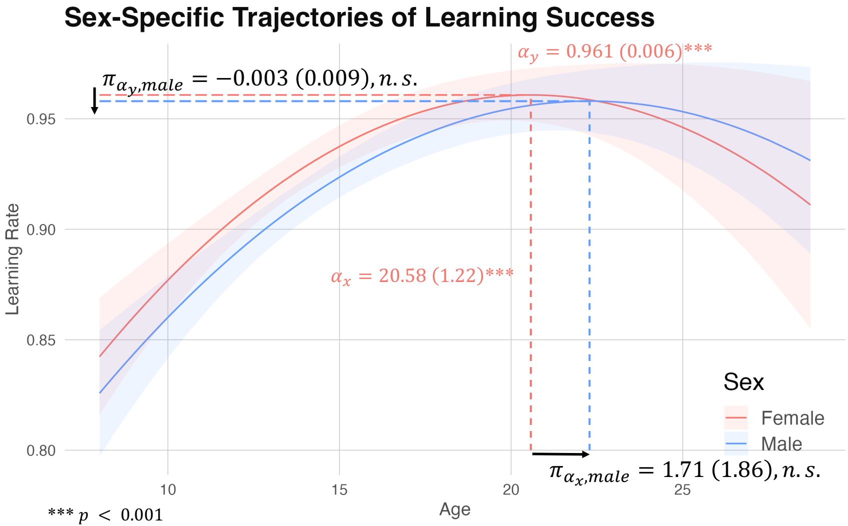
LENI Approach to Fixed Effects Estimation

Pop. θ		Linear Estimates		LENI Estimates		Nonlinear Estimates	
Cubic Model							
x_N	0	β_0	9.994 (0.139)	x_N	−0.002 (0.116)	x_N	−0.002 (0.116)
y_N	10	β_1	−1.000 (0.080)	y_N	9.996 (0.094)	y_N	9.996 (0.094)
δ	3	β_2	2.25×10^{-4} (0.013)	δ	3.015 (0.107)	δ	3.015 (0.107)
h	−2	β_3	0.037 (0.005)	h	−2.010 (0.130)	h	−2.010 (0.130)
R^2	0.5		0.506				
BIC			921.90				921.90

| Sex-Specific Delays in Peak Learning? Perhaps not



$^{***}p < 0.001$

Linearized SEM: Polynomials and Beyond

LENI approach can be applied to mixed-effects polynomial models [6]

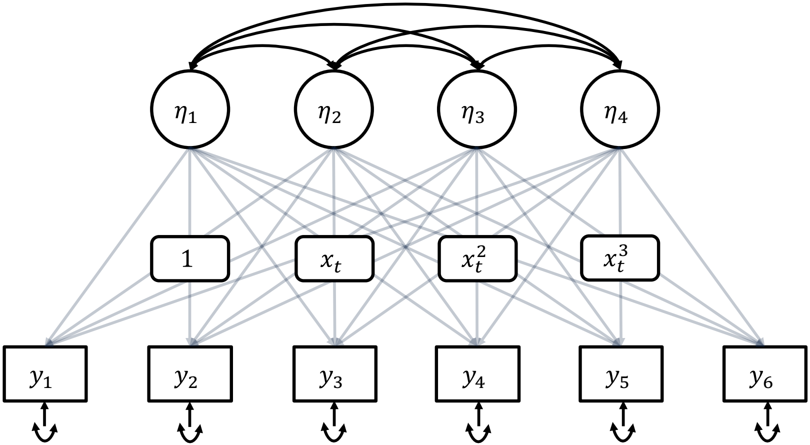
- equivalence between the curves should result in identical fit (assuming convergence)
- defined transformations can be applied to the linear fitted model, including fixed effects, random effects*, and conditional effects of covariates
- limited functions with linear and nonlinear versions

Structured latent curve model [2, 1, 9] approach can offer additional flexibility and target functions

- Does not require a linearly equivalent form
- Uses a Taylor series approach for linearized approximation

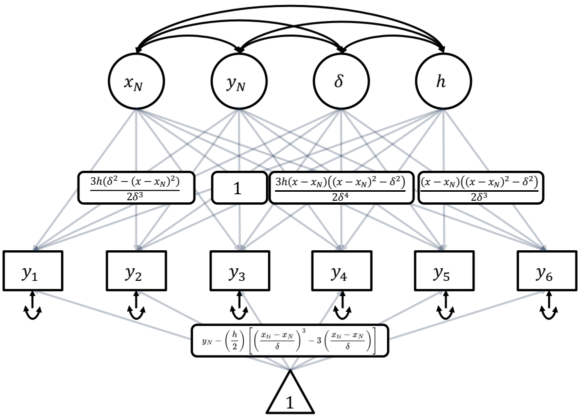
| Standard Cubic Growth Model

$$y_{ti} = \beta_0 + \beta_1 x_{ti} + \beta_2 x_{ti}^2 + \beta_3 x_{ti}^3$$

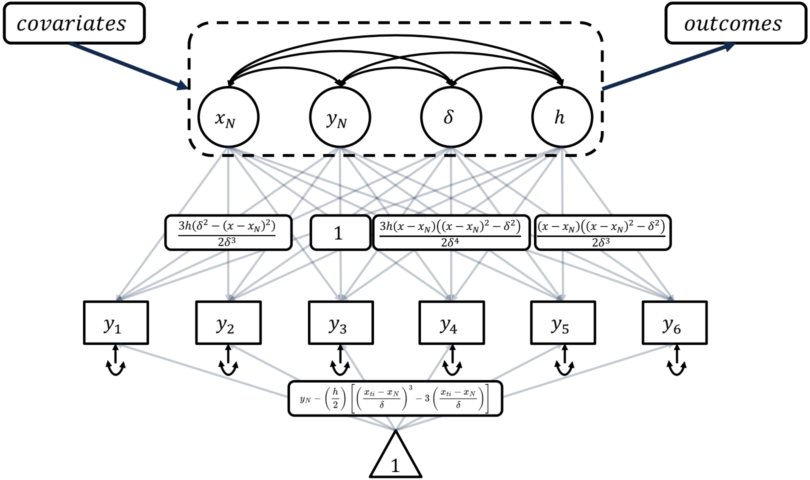


| Alternative Cubic Growth Model

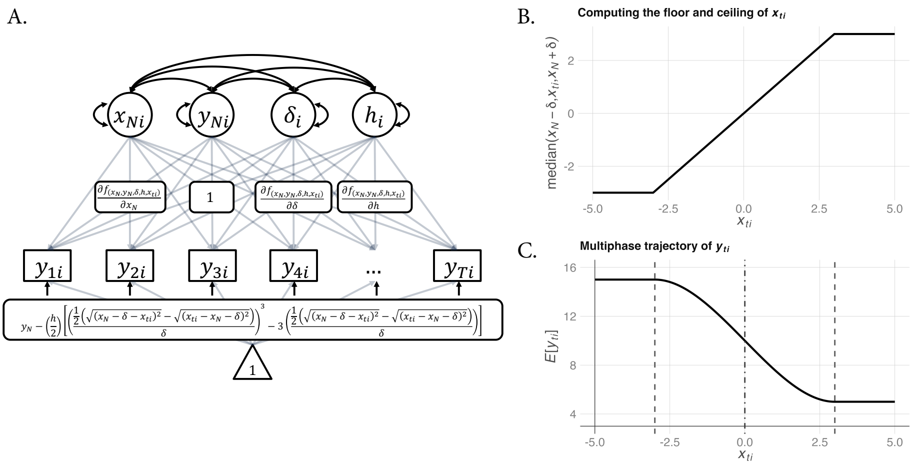
$$y_{ti} = y_N - \frac{h}{2}\left[\left(\frac{x_{ti}-x_N}{\delta}\right)^3 - 3\left(\frac{x_{ti}-x_N}{\delta}\right)\right]$$



| Extending the SLCM to include Covariates and Distal Outcomes

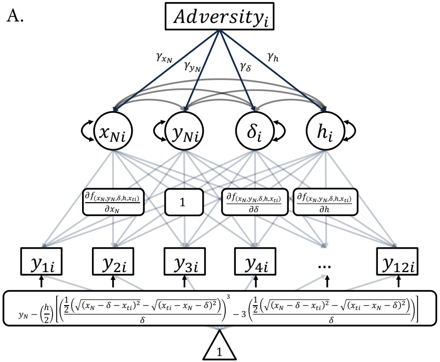


| SCLM for the Multiphase Cubic



| Adversity-informed Trajectories of Cortical Thinning

A.



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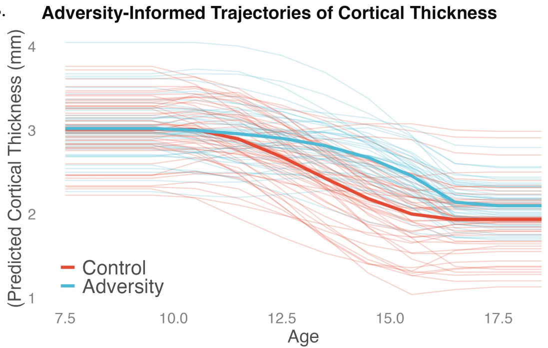

What's Does this Allow Us to Do

Aligning statistical models with research hypotheses is a challenge, but nonlinear models can help.

- Challenges of estimation can be solved through linear estimation and transformation.
- Polynomial models have many options, while fully nonlinear models require moving to SEM (for now)

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Collaboration and future work

- If you see opportunities for nonlinear models in your own area of work, please get in touch
- AMPPS tutorial paper
- Looking to hire a student for Fall 2025 to continue this and other methodological work at the University of Delaware

Questions?

Ethan M. McCormick
Department of Methodology & Statistics
e.m.mccormick@fsw.leidenuniv.nl

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